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A quantum Hall fluid of vortices

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Abstract: In this note we demonstrate that vortices in a non-relativistic Chern-Simons theory form a quantum Hall fluid. We show that the vortex dynamics is controlled by the matrix mechanics previously proposed by Polychronakos as a description of the quantum Hall droplet. As the number of vortices becomes large, they fill the plane and a hydrodynamic treatment becomes possible, resulting in the non-commutative theory of Susskind. Key to the story is the recent D-brane realisation of vortices and their moduli spaces.

Keywords: Solitons Monopoles and Instantons, Chern-Simons Theories, Brane Dynamics in Gauge Theories.

Contents

1. The introduction

Chern-Simons theories [1] provide an effective, long distance description of the fractional quantum Hall effect (FQHE). In fact, they provide several such descriptions. The range of models on the market fall roughly into one of two categories depending on the physical interpretation of the vector potential A appearing in the $A \wedge F$ term. In the initial papers on the subject [2, 3], A acts as a statistical gauge field of the type suggested by Wilczek [4]. Its role is to endow the excitations of the model with the charge and statistics appropriate to the quantum Hall system. Later works concentrate on hydrodynamic properties of the quantum Hall fluid in which either A or ${}^\star F$ are vector fields associated with conserved currents and charge density [5, 6].[1]

More recently, Susskind has suggested that the hydrodynamic properties of the quantum Hall fluid are captured by a Chern-Simons theory at level k, defined on a non-commutative background [9]. The electrons sit at Laughlin filling fraction $\nu = 1/(k+1)$ and the fluid fills the infinite plane. Subsequently, Polychronakos proposed a matrix model regularisation of Susskind's theory in order to describe a finite quantum Hall droplet consisting of N electrons [10]. As $N \to \infty$, the droplet expands to fill the plane and we recover Susskind's non-commutative dynamics. Several properties of this matrix model have since been explored, including the relationship to Laughlin wavefunctions [11, 12] and the coupling to external electromagnetic fields [13].

In this paper we study a non-relativistic Chern-Simons theory defined on an ordinary, mundane space in which the coordinates commute. The theory does not give an immediate description of a fractional quantum Hall fluid, but rather defines a background into which spin-polarised (i.e. spinless) electrons may be injected. These electrons arise as the vortices

[1] For readers whose brane activity usually takes place at the Planck scale, introductions to facets of the FQHE may be found in [2, 7, 8].

of the theory and we show that their quantum dynamics is controlled by the matrix model of Polychronakos. The vortices thus form a fractional quantum Hall droplet. As the number of vortices becomes large, they may be described by Susskind's hydrodynamic, non-commutative Chern-Simons theory.

The key to the connection between vortex dynamics and the FQHE is provided by the recent string theory realisation of vortices and their moduli spaces given in [14]. While [14] considered vortices in the relativistic Maxwell-Higgs theory, we here extend the results to the non-relativistic Chern-Simons case. Rather than present a new D-brane picture, we instead make use of known connections between vortex dynamics in Maxwell and Chern-Simons theories [15, 16]. It would be interesting to relate the results presented here to the D-brane constructions of [17, 18].

The observation that, under favourable conditions, vortices may form a quantum Hall fluid is hardly new. It is implicit in the hierarchy construction of FQH states through the condensation of quasiparticles. In the context of superconductivity, it was first suggested by Stern [19], motivated by the similarity between the Magnus force and the Lorentz force. Recently the idea has received much attention in the context of rotating Bose systems — see for example [20]. Here we give a simple, string theory inspired, derivation of this effect.

2. The vortex

The non-relativistic model that we consider consists of a single complex scalar field ϕ, coupled to a U(1) gauge field a_μ,

$$L = \int d^2x \;\; i\phi^\dagger \mathcal{D}_0\phi - \frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - \mu a_0 - \Delta|\mathcal{D}_i\phi|^2 - \frac{2\pi\Delta}{k}\left(|\phi|^2 - \mu\right)^2 \,, \tag{2.1}$$

where the covariant derivative is given by $\mathcal{D}_\mu = \partial_\mu - ia_\mu$. The theory was previously proposed by Manton [15] as a non-dissipative model for vortex motion in superconductors.[2] Here a_μ will play the role of a statistical gauge field. The equation of motion for a_0 yields Gauss' law,

$$b = \epsilon^{ij}\partial_i a_j = \frac{2\pi}{k}\left(|\phi|^2 - \mu\right) . \tag{2.2}$$

The chemical potential term μa_0 ensures that the potential energy can be minimised by $|\phi|^2 = \mu$ with $b = 0$. The theory lives in a gapped phase with broken gauge symmetry, and therefore admits topologically stable vortices with winding number $N \in \mathbb{Z}$

$$\int d^2x \; b = -2\pi N \,. \tag{2.3}$$

The coefficients in (2.1) are not arbitrary. Firstly, we require that the Chern-Simons level is quantised: $k \in \mathbb{Z}$. We pick $k > 0$. Secondly, the coefficients in the potential energy have been fine-tuned so that the second order equations of motion may be integrated once [21]. It can be checked that, for winding number $N > 0$, the equations of motion are satisfied by solutions to (2.2) together with the first order equation,

$$\mathcal{D}_z\phi = 0\,, \tag{2.4}$$

[2]Strictly speaking, the lagrangian agrees with that of [15] only after imposing Gauss' law.

where we have defined the complex coordinate $z = x^1 + ix^2$ and $a_z = \frac{1}{2}(a_1 - ia_2)$. The energy required to excite N such vortices is $\mathcal{E} = 2\pi\mu N\Delta$. This formula contains no hint of binding energy and, indeed, it can be shown that that there are no static forces between the vortices. Note that because equation (2.2) comes from Gauss' law, vortices with this property exist only for $N > 0$.

Although our starting point was a non-relativistic Chern-Simons theory, the vortex equations (2.2)–(2.4) coincide with those arising in the relativistic Maxwell-Higgs model. The solutions to these equations therefore also describe vortices in a critically coupled superconductor (i.e. on the borderline between type I and type II). The fact that the same vortices are shared by the non-relativistic Chern-Simons theory and the relativistic Maxwell theory will prove crucial in the following.

While no analytic solutions to the vortex equations are known, index theorems reveal that the most general solution contains $2N$ parameters [22]. These may be taken to be unordered N-tuple of positions z^a, $a = 1, \ldots, N$ on the complex plane, each of which corresponds to a zero of the Higgs field. The *moduli space* of vortices, defined as the space of solutions to the vortex equations, is therefore a $2N$-dimensional manifold which we shall denote as $\mathcal{M}_N$. Geometrically, $\mathcal{M}_N \cong \mathbb{C}^N/S_N$, where S_N is the permutation group of N elements, reflecting the fact that the vortices are indistinguishable. In the asymptotic region of $\mathcal{M}_N$, when $|z^a - z^b|$ is larger than all other length scales, the solution looks like N well-separated vortices, each containing a single quantum of flux. However, as the vortices approach, the orbifold singularities of $\mathbb{C}^N/S_N$ are smoothed out. At this point the z^a are no longer good coordinates and one should transform to another basis in which $\mathcal{M}_N$ is manifestly smooth. The purpose of this paper is to show that in this regime, as the vortices approach, they form a quantum Hall fluid.

3. The dynamics

The lagrangian (2.1) was chosen so that there are no static forces between vortices. In a derivative expansion, the velocity dependent interactions are therefore dominant. For slow moving vortices, these may be elegantly captured using the Manton moduli space approximation. This assumes that all time dependence is restricted to the collective coordinates $z^a = z^a(t)$. Substituting the time dependent configurations into the kinetic terms of (2.1) then gives rise an effective quantum mechanics for z^a.

Let us first recall the story for vortices in the relativistic Maxwell-Higgs model [23], since this situation will turn out to be intimately woven with our own. Here the kinetic terms are second order and $\mathcal{M}_N$ is understood as the *configuration space* of the vortex system. The moduli space approximation defines a Kähler metric g on $\mathcal{M}_N$ which captures the low-energy energy dynamics,

$$\mathcal{L}_{\mathrm{Maxwell}} = \frac{1}{2} g_{ab}(z^c, \bar{z}^c)\, \dot{z}^a \dot{\bar{z}}^b \,. \tag{3.1}$$

The metric g is constructed in such a way that the geodesics track the classical scattering of vortices.

In the present case, the kinetic terms in our non-relativistic lagrangian are first order and $\mathcal{M}_N$ now plays the role of the *phase space* of the vortex system. The low-energy dynamics of the vortices is of the form,

$$L_{\rm CS} = \frac{i}{2}\left(\bar{f}_a(z^b,\bar{z}^b)\,\dot{z}^a - f_a(z^b,\bar{z}^b)\,\dot{\bar{z}}^a\right), \tag{3.2}$$

where $\mathcal{A} = \bar{f}_a dz^a - f_a d\bar{z}^a$ is a connection on $\mathcal{M}_N$. The task of determining $\mathcal{A}$ was undertaken by Manton [15] and Romão [16]. For far separated vortices, they show that $\bar{f}_a \to \pi\mu\bar{z}^a$ which simply describes non-interacting fluxes in the condensate μ. In this regime, the lagrangian becomes equivalent to one describing non-interacting electric charges in a large magnetic field $B = 2\pi\mu$, providing a dual picture to which we shall return later. For the purposes of this paper we are more interested in the physics when the vortices approach. Here an explicit expression for $\mathcal{A}$ is not known. However, it can be shown that $\mathcal{A}$ has the simple geometrical interpretation [15, 16]

$$d\mathcal{A} = -i\Omega\,, \tag{3.3}$$

where Ω is the Kähler form with respect to the metric g on $\mathcal{M}_N$. This result provides a connection between the dynamics of vortices in the Chern-Simons theory and the dynamics of vortices in the Maxwell theory, and will play an important role in the following section. However, it is not of immediate use in determining the physics of closely packed vortices. The trouble lies in the fact that, like $\mathcal{A}$, little is known about the metric g. In the asymptotic regime $|z^a - z^b| \gg 1$ the metric becomes flat, once again reflecting the fact that far-seperated vortices may be thought of as non-interacting particles. To make progress in understanding the dynamics in the limit in which the vortices approach, we turn to string theory for inspiration.

4. The matrix

Let us start once more with vortices in the relativistic Maxwell-Higgs theory. Recently, a D-brane construction of this model was given in type IIB string theory [14]. In this set-up, the vortices appear as D-strings suspended between NS5-branes and D3-branes, and their dynamics can be easily determined. Let us quickly review the main result. It was found that the dynamics of N D-strings is encoded in a $\mathrm{U}(N)$ gauged quantum mechanics, containing a complex matrix $Z^a_{\ b}$, $a,b = 1,\ldots,N$ transforming in the adjoint of $\mathrm{U}(N)$, and a complex vector ψ^a transforming in the fundamental representation. The low-energy dynamics of the D-strings is given by.[3]

$$\mathcal{L}_{\rm D-brane} = \mathrm{Tr}\left(\pi\mu\,\mathcal{D}_t Z^\dagger \mathcal{D}_t Z - \lambda^2\left(\psi\psi^\dagger + \pi\mu\,[Z,Z^\dagger] - \kappa\right)^2\right) + \mathcal{D}_t\psi^\dagger\mathcal{D}_t\psi\,. \tag{4.1}$$

Here $\mathcal{D}_t Z = \dot{Z} - i[A_0, Z]$ and $\mathcal{D}_t\psi = \dot{\psi} - iA_0\psi$ where A_0 is a vector potential which may be completely gauged away. String theory instructs us to take the $\lambda^2 \to \infty$ limit, imposing

[3] We have rescaled Z by the vortex mass relative to [14] so that it has the correct dimension. To compare with the conventions of [14], note that $\zeta \equiv \mu$ and $e^2 \equiv 2\pi/\kappa$.

the N^2 constraints

$$\psi^a \psi^\dagger_b + \pi\mu\, [Z, Z^\dagger]^a{}_b = \kappa \delta^a{}_b \tag{4.2}$$

on the $2N(N+1)$ degrees of freedom contained within Z and ψ. Restricting to $\mathrm{U}(N)$ invariant objects as required by the gauge symmetry imposes a further N^2 constraints, leaving a remaining $2N$ degrees of freedom. These describe the positions of the ends of N D-strings moving on the plane. Since the D-strings are identified with vortices, these $2N$ degrees of freedom given give natural coordinates on the moduli space $\mathcal{M}_N$.

The details of the classical D-brane dynamics described by the matrix model (4.1) do not coincide with the vortex dynamics described by the moduli space metric (3.1). Nevertheless, the matrix mechanics does capture many of the qualitative features of the vortices, including the symmetries, singularity structure and scale of the moduli space. Moreover, when attention is restricted to certain "topological" or "BPS" quantum correlation functions in supersymmetric theories, one can replace the true vortex dynamics (3.1) with the D-brane dynamics (4.1) and obtain quantitatively correct answers — see [14] for further discussions.

In this paper we shall describe the dynamics of vortices in the Chern-Simons theory in a similar matrix fashion. Without supersymmetry as our guardian, it is hard to rigorously justify this step. Nevertheless, we continue forward under the assumption that the matrix mechanics correctly captures the relevant qualitative features of the vortex moduli space. Given the conclusions of this paper, it would be interesting to return to the moduli space description (3.2), perhaps using the geometric quantisation techniques propounded in [16], in an attempt to reproduce the results without resorting to string theory.

So, if the matrix model (4.1) describes the dynamics of vortices in the Maxwell theory, what is the relevant matrix model to describe the dynamics of vortices in our Chern-Simons theory? The answer lies in the relationship $d\mathcal{A} = -i\Omega$ which relates the vortex dynamics in the two theories. We must simply ensure that our two matrix models obey a similar relationship. To do this, we first need an expression for the counterpart of Ω in the matrix model. In fact, this is rather simple since the matrix model is constructed in such a way that the Kähler form on $\mathcal{M}_N$ is inherited from the canonical Kähler form on the unconstrained space parameterised by Z and ψ. This process, known as the symplectic quotient construction, ensures that we may work with the obvious first order system using the variables Z and ψ,

$$\mathcal{L} = i\pi\mu\, \mathrm{Tr}\left(Z^\dagger \dot{Z}\right) + i\psi^\dagger \dot{\psi} \tag{4.3}$$

and subsequently restrict to the moduli space $\mathcal{M}_N$ defined by $\mathrm{U}(N)$ invariant observables subject to the constraint (4.2). This latter step may be achieved by re-introducing A_0, now playing the role of a Lagrange multiplier. The low-energy dynamics of the vortices may therefore be described by the matrix mechanics

$$\mathcal{L}_{\mathrm{matrix}} = \mathrm{Tr}\left(i\pi\mu\, Z^\dagger \mathcal{D}_t Z - \kappa A_0\right) + i\psi^\dagger \mathcal{D}_t \psi\,. \tag{4.4}$$

This expression, describing the dynamics of Chern-Simons vortices, is the main result of this paper.

5. The Hall fluid

The matrix model (4.4) was previously proposed by Polychronakos as a description of N electrons moving in the lowest Landau level of a background magnetic field $B = 2\pi\mu$ [10]. The electrons are identified with our vortices, and from now on we treat the terms synonymously. The classical and quantum dynamics arising from the matrix model have been studied in great detail (see [10] and references therein). Here we mention a few choice details. Most pertinently, it can be shown that when the electrons coalesce, they manifest the properties of a quantum Hall fluid of density ρ where

$$B = 2\pi\mu\,, \qquad \rho = \frac{\mu}{k}\,.$$

This gives rise to a classical filling fraction $\nu = 2\pi\rho/B = 1/k$. In fact, there is an important quantum shift pointed out in [10] (see also [18]) so that the system actually describes a Hall fluid at filling fraction $\nu = 1/(k+1)$.

Let us try to understand this behaviour from the perspective of critically coupled vortices. At first glance it seems peculiar that the vortex dynamics would give rise to a FQH fluid since the lagrangian (3.2) contains no sign of the repulsive particle interactions that are usually held accountable for such an effect. Indeed, the hamiltonian associated to (3.2) vanishes and, for $|z^a - z^b|$ suitably large, the solution to the vortex equations can be understood as N far-separated, non-interacting vortices. Each has size $L \sim \sqrt{k/\mu}$ which (ignoring factors of 2 and π) is the penetration depth in the language of superconductivity. In this paper we are interested in the the situation with $|z^a - z^b| < L$. What do the vortex configurations look like in this regime? We suggest that the vortices should not be thought of as overlapping particles, but rather as a *classically* incompressible fluid whose density remains constant at L^{-2} for all values of $|z^a - z^b| < L$. To see that this gives rise to a consistent picture, note that the vortices see a background condensate μ which, as we have seen, can be thought of as a background magnetic flux for charged particles in a dual picture. The density of vortex states required to fill the "dual Landau level" is therefore $\sim \mu$. With the vortices at a density of L^{-2}, this gives rise to the required filling fraction $\nu \sim 1/\mu L^2 \sim 1/k$. Clearly the speculations offered in this paragraph refer to properties of the classical vortex solutions, and it is to be hoped that they can be confirmed (or dismissed) by an explicit study of the vortex equations. A discussion of the classical incompressible vortex liquid and its interpretation in tems of exclusion statistics can be found in [24].

Finally, recall that as the number of electrons/vortices becomes large and $N \to \infty$, the constant term in the constraint (4.2) may be absorbed by the commutator rather than the $\psi\psi^\dagger$ term,

$$[Z, Z^\dagger] = \frac{k}{\pi\mu} \equiv 2\theta\,. \tag{5.1}$$

Expanding around this background, the matrix model (4.4) may be re-written as a U(1) Chern-Simons theory at level k defined on the plane with the coordinates satisfying (5.1). This is Susskind's hydrodynamic description of the FQHE [9]. It is amusing that, having started with a commutative U(1) Chern-Simons-Higgs theory at level k, we return via vortex dynamics to a non-commutative U(1) Chern-Simons theory at level k.

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6. The potential

As it stands, there is nothing to keep the electrons in (4.4) from wandering over the plane. When the electrons coalesce they form a FQH fluid, but when they sit far apart they return to their individual, yet indistinguishable, electronic identities. In order to energetically distinguish these two scenarios and coax the electrons together, Polychronakos introduced a simple harmonic oscillator potential [10] whose role is to trap the electrons close to the origin,

$$V = \frac{Bw}{2}\,\mathrm{Tr}\left(Z^\dagger Z\right)\,. \tag{6.1}$$

In this section, we will see how to generate such a potential for the vortex dynamics of the Chern-Simons theory. First note that if our only requirement is to provide a rotationally symmetric potential which will be seen by the vortices and pen them near the origin, then one could simply add to (2.1) a term of the form

$$V_0 = \frac{Bw}{2}\int d^2x\ |z|^2\left(\frac{k}{2\pi}b^2 + |\mathcal{D}_i\phi|^2\right).$$

However, if we want to match to the energetics of [10], then it is possible to provide the deformation that gives rise to the harmonic oscillator potential (6.1). The key observation is that equation (6.1) is a mass term for Z which induces a potential on the moduli space $\mathcal{M}_N$ that is (up to an unimportant constant) proportional to the norm-squared of the Killing vector associated to rotational symmetries. Such potentials appear frequently in soliton dynamics and can be written as the overlap of the corresponding zero modes of the soliton using the method of [25]. Here we omit the details (mostly associated with gauge fixing the zero mode) and simply state the result: the potential (6.1) for the vortex dynamics is generated by augmenting the Chern-Simons lagrangian with the potential

$$V = V_0 + \frac{Bwk}{4\pi}\int d^2x\ \frac{1}{2}(\partial_i\Lambda)^2 + \frac{2\pi}{k}\Lambda^2|\phi|^2 - 2\Lambda b\,.$$

Here the function Λ arises when fixing the gauge for the vortex zero mode and is to be evaluated on the solution to its classical equation of motion in the background of the vortex.

7. The end

Let us mention a few generalisations of the story. The lagrangian of our Chern-Simons theory (2.1) was fine-tuned to ensure that the vortices experience no static force. It is natural to wonder what happens if this is no longer the case. For example, we may change the coefficient of the potential term in (2.1) by adding,

$$\Delta V = \gamma\int d^2x\ \left(|\phi|^2 - \mu\right)^2\,.$$

Then for $\gamma < 0$, the vortices attract (type I superconductivity), while for $\gamma > 0$, the vortices repel (type II). In this latter case, the repulsive force competes with the harmonic oscillator potential (6.1) which pushes the vortices towards the origin. For small γ, we expect the

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quantum Hall state to persist. In contrast, for suitably large γ the vortices will undergo a phase transition to the more familiar Abrikosov lattice or, in the dual language of electrons, the Wigner crystal.

The D-brane construction of [14] provides several further generalisations, including non-abelian Chern-Simons terms, extra scalar fields, and non-commutative backgrounds. For example, one could consider vortices in U(m) Chern-Simons-Higgs theory. The low-energy dynamics of these vortices is described by the matrix model (4.4), now with m vectors ψ. This model describes m quantum Hall layers and was previously studied in [26]. As the number of vortices becomes large, it reduces to U(m) non-commutative Chern-Simons theory.

To summarise, we have shown that the fractional quantum Hall matrix model of Polychronakos [10] can be thought of as describing the low-energy dynamics of vortices in a non-relativistic Chern-Simons theory. We suggest that the physical reason for this behaviour is the classically incompressible nature of vortices as they coalesce. A crucial ingredient in our story was the D-brane construction of [14] and, in the absence of a field theory derivation, the quantum Hall fluid of critically coupled vortices can be taken as a prediction of string theory. It is to be hoped that this new perspective on the quantum Hall matrix model may help in building the dictionary to physical quantities such as currents, particle density and the Laughlin wavefunctions.

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